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Koszul Complexes, Differential Operators, and the Weil–Tate Reciprocity Law

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0. INTRODUCTION

The Weil–Tate multiplicative reciprocity law on an algebraic curve [W] is an essential tool in the abelian class field theory of its function field. In this paper we study the geometry of this reciprocity law in connection with Koszul complexes and with the Krichever map. The Weil–Tate reciprocity law deals with two meromorphic functions f, g on a projective algebraic curve X. It is enough to write it down when f and g are holomorphic outside some smooth point a. Then we can assume that f and g are normalized with respect to a local parameter z at a, so that their dominant terms are z^{-n} and z^{-m}, respectively.

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Now we can write the reciprocity law as

$$\prod_{x\in X\setminus\{a\}} f(x)^{v_x(g)} = (-1)^{mn} \prod_{x\in X\setminus\{a\}} g(x)^{v_x(f)}. \tag{0.1}$$

We give a direct algebraic proof of (0.1) in the first section, based on the fact that the left-hand side is $\det(f, A/gA)$, where $A = H^0(X\setminus\{a\}, \mathcal{O}_X)$. This can be described as the determinant of the Koszul double complex for f and g acting on A. The equality (0.1) then amounts to the fact that the Koszul complex is essentially invariant under permutation of f and g, up to a sign that we compute. In order to deal with infinite-dimensional determinants, we filter A by the order of pole at a, and study finite-dimensional determinants.

The second section is devoted to relating this Koszul complex approach to the proof of (0.1) given by the second author in [P], using the differential operators occurring in Krichever's construction. We recall this construction, which (given a divisor on X in general position) embeds A into the algebra of holomorphic differential operators in a variable x. In Section 2 we give three different points of view on the Krichever construction; the approach via the Nakayashiki–Mukai Fourier transform is emphasized because of its potential applications to reciprocity laws in higher dimensions.

Finally, Section 3 explains the relation between the two approaches, using a duality between A/fA and the space of solutions of the corresponding differential operator L_f.

We view this work as the first part of a program for understanding multiplicative reciprocity laws in terms of Koszul complexes and partial differential operators in higher dimensions. We believe the work of Nakayashiki [N1, N2] will play a crucial role there.

1. KOSZUL COMPLEXES AND THE WEIL RECIPROCITY LAW

First we derive an equality of determinants (Prop. 1.2) in a linear-algebra setup, then we apply it to a geometric context.

For any vector space E of dimension m, we have the line $\wedge^{\max}E$. Any basis $\bar{a} = (a_1, \ldots, a_m)$ of E induces an element $\wedge\bar{a} = a_1 \wedge \cdots \wedge a_m$ of $\wedge^{\max}E$.

Given an exact sequence

$$0 \to E \xrightarrow{u} F \to K \to 0,$$

we define an isomorphism $\alpha\colon \wedge^{\max}E \otimes \wedge^{\max}K \tilde{\to} \wedge^{\max} F$ as follows: let $(a_1, \ldots, a_m)$ be a basis of E and $(b_1, \ldots, b_p)$ be a basis of K. Pick lifts $\bar{b}_i$ of b_i in F. Then we set

$$\alpha(\wedge\bar{a} \otimes \wedge\bar{b}) = u(a_1) \wedge \cdots \wedge u(a_m) \wedge \bar{b}_1 \wedge \cdots \wedge \bar{b}_p. \tag{1.1}$$

We also denote by α the corresponding isomorphism $\wedge^{\max} E \tilde{\to} \wedge^{\max} F \otimes \wedge^{\max} K^{\otimes -1}$. In particular, the exact sequence

$$0 \to E \to E \oplus K \to K \to 0$$

induces

$$\alpha \colon \wedge^{\max} E \otimes \wedge^{\max} K \tilde{\to} \wedge^{\max} (E \oplus K).$$

We note the commutative diagram

$$\begin{array}{ccc} \wedge^{\max} E \otimes \wedge^{\max} K & \tilde{\to} & \wedge^{\max}(E \oplus K) \\ \Big\downarrow \sigma & & \Big\downarrow (-1)^{mp} \\ \wedge^{\max} K \otimes \wedge^{\max} E & \tilde{\to} & \wedge^{\max}(E \oplus K), \end{array}$$

where σ is the permutation of the two factors.

Now consider a commutative diagram of finite dimensional vector spaces with exact rows

$$\begin{array}{ccccccccc} 0 & \to & E & \xrightarrow{u} & F & \to & K & \to & 0 \\ & & \Big\downarrow w & & \Big\downarrow v & & \Big\downarrow y & & \\ 0 & \to & G & \xrightarrow{x} & H & \to & L & \to & 0. \end{array} \tag{1.2}$$

We assume that y is an isomorphism. Say we have

$$\dim(E) = m, \dim(F) = m + p, \dim(G) = m + q, \dim(H) = m + p + q.$$

We construct an isomorphism

$$\psi : \wedge^{\max} E \otimes \wedge^{\max} H \tilde{\to} \wedge^{\max} F \otimes \wedge^{\max} G \tag{1.3}$$

by composing three maps, namely

$$\alpha \otimes \alpha \colon \wedge^{\max} E \otimes \wedge^{\max} H \to [\wedge^{\max} F \otimes \wedge^{\max} K^{\otimes -1}] \otimes [\wedge^{\max} G \otimes \wedge^{\max} L],$$

the permutation of the factors $\wedge^{\max} G$ and $\wedge^{\max} L$, and the isomorphism

$$\wedge^{\max} F \otimes [\wedge^{\max} K^{\otimes -1} \otimes \wedge^{\max} L] \otimes \wedge^{\max} G \tilde{\to} \wedge^{\max} F \otimes \wedge^{\max} G$$

induced by the isomorphism $\wedge^{\max} y \colon \wedge^{\max} K \tilde{\to} \wedge^{\max} L$.

We can then compute ψ as follows. Let $(c_1, ..., c_q)$ be vectors in G such that $(w(a_1), ..., w(a_m), c_1, ..., c_q)$ forms a basis of G. Let $(d_1, ..., d_p)$ be a basis of L. Pick lifts $\bar{d}_i$ of these vectors to H. Then we have

$$\begin{aligned} &\psi\ (\wedge \bar{a} \otimes [xw(a_1) \wedge \cdots \wedge xw(a_m) \wedge x(c_1) \wedge \cdots \wedge x(c_q) \wedge \bar{d}_1 \wedge \cdots \wedge \bar{d}_p]) \\ &= \det(y)[\wedge \bar{a} \wedge \bar{b}_1 \wedge \cdots \wedge \bar{b}_p], \end{aligned}$$

where the determinant is computed in the bases $b_1, \ldots, b_p$ and $d_1, \ldots, d_p$.

On the other hand, we can form the complex

$$0 \to E \overset{(u,-w)}{\longrightarrow} F \oplus G \overset{(v,x)}{\longrightarrow} H \to 0. \tag{1.4}$$

This is an exact sequence, because y is an isomorphism. The complex (1.4) induces an isomorphism

$$\phi \colon \wedge^{\max} E \otimes \wedge^{\max} H \overset{\alpha}{\longrightarrow} \wedge^{\max} (F \oplus G) \tilde{\to} \wedge^{\max} F \otimes \wedge^{\max} G. \tag{1.5}$$

We note

LEMMA 1.1. $\phi = (-1)^{p(q+m)} \psi$.

Now let ω denote the analog of the isomorphism ψ for the diagram

$$\begin{array}{ccc} E & \to & G \\ \downarrow & & \downarrow \\ F & \to & H, \end{array} \tag{1.6}$$

where the rows and columns in (1.2) have been permuted. This permutation leads to a new complex

$$0 \to E \to G \oplus F \to H \to 0, \tag{1.7}$$

and to an isomorphism $\gamma \colon \wedge^{\max} E \otimes \wedge^{\max} H \tilde{\to} \wedge^{\max} G \otimes \wedge^{\max} F$. We then have

$$\gamma = (-1)^{m+(p+m)(q+m)} \sigma \phi, \tag{1.8}$$

where σ is the permutation of the factors $\wedge^{\max} F$, $\wedge^{\max} G$.

Now we have from Lemma 1.1,

$$\omega = (-1)^{q(p+m)} \gamma. \tag{1.9}$$

It then follows that we have

PROPOSITION 1.2. $\omega = (-1)^{pq} \sigma \psi$.

The sign here is the product of the signs in Lemma 1.1, (1.8) and (1.9).

Note that if we designated bases in our vector spaces, then we can view $\psi, \phi, \omega, \gamma$ as numbers.

We apply this in the following situation. Let A be the algebra of regular functions on $X \min\{a\}$, where X is a smooth projective algebraic curve. Then we have a filtration of A by the subspaces $A_m = \{f \in A, v_a(f) + m + g - 1 \geq 0\}$ which are of dimension m for m large enough. Let z be a local parameter at a. We can define bases in A_m as follows. Let $n_1 < n_2 < \cdots < n_g$ be the Weierstrass gaps at a. The complement subset Γ is the Weierstrass semigroup. Let $r_1 < r_2 < \cdots < r_m$ be the elements of

$\{0, ..., m+g-1\} \cap \Gamma$. Then a basis $(a_1, ..., a_m)$ of A_m is called normalized if a_j has a leading term z^{-r_j}. Such a_j is also said to be normalized.

We consider P and Q elements of A of order of pole p, q and assume they too are normalized. We consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & A_m & \xrightarrow{P} & A_{m+p} & \to & A/PA & \to & 0 \\ & & \big\downarrow{\scriptstyle Q} & & \big\downarrow{\scriptstyle Q} & & \big\downarrow{\scriptstyle Q} & & \\ 0 & \to & A_{m+q} & \xrightarrow{P} & A_{m+p+q} & \to & A/PA & \to & 0. \end{array} \tag{1.10}$$

We assume that P and Q have no common zero in $X \min\{a\}$, or equivalently $A = PA + QA$. Then Q induces an isomorphism $A/PA \to A/PA$.

The complex (1.4) then becomes the complex

$$0 \to A_m \overset{(P,-Q)}{\longrightarrow} A_{m+p} \oplus A_{m+q} \overset{(Q,P)}{\longrightarrow} A_{m+p+q} \to 0 \tag{1.11}$$

which is a subcomplex of the Koszul complex on P, Q obtained by bounding the order of the terms. It is quasi-isomorphic to the Koszul complex (in fact, both are exact).

We next specify a normalized type of basis on A/PA. Let $s_1 < \cdots < s_p$ be the elements of Γ which are $\leq m+p+g-1$ and such that they do not belong to $p+\Gamma$. Then any p-tuple $(b_1, \ldots, b_p)$ of normalized elements of A of orders $-s_1, \ldots, -s_p$ will give a basis of A/PA. Such a basis will be called normalized. We verify easily that the isomorphism

$$\alpha\colon \wedge^{\max} A_m \otimes \wedge^{\max}[A/PA] \tilde{\to} \wedge^{\max} A_{m+p} \tag{1.12}$$

transforms the tensor product of two normalized bases into ϵ_m times a normalized basis, where ϵ_m is the sign of the permutation of $\Gamma_{m+p} = \Gamma \cap \{0, \ldots, g+m+p-1\}$ which first lists in order the elements of $p+\Gamma_m$, then lists in order the remaining elements.

We do not have a general formula for ϵ_m, but we have

LEMMA 1.3. $\epsilon_m \epsilon_{m+q} = (-1)^{pq}$.

With all our normalized bases, we can view ψ, ϕ, ω, γ as numbers. Then we obtain from Lemma 1.3,

$$\psi = (-1)^{pq} \det(Q, A/PA). \tag{1.13}$$

Similarly we have

$$\omega = (-1)^{pq} \det(P, A/QA). \tag{1.14}$$

So we get an equivalent form of the Weil reciprocity law (0.1).

PROPOSITION 1.4. *For two elements P, Q in A with $v_a(P) = p, v_a(Q) = q$, we have*

$$\det(P, A/QA) = (-1)^{pq} \det(Q, A/PA).$$

Now we have

$$\det(P, A/QA) = \prod_{y \in X \min\{a\}} P(y)^{v_y(Q)}. \tag{1.15}$$

Therefore we recover the usual form of the Weil reciprocity law.

We briefly discuss the relation with the classical resultant of polynomials P, Q in x of degrees p, q. In this case $A = k[x]$ and A_m is the space of polynomials of degree $\leq m-1$. We use $z = x^{-1}$ as the parameter at $\infty \in \mathbb{P}^1$. We have the normalized basis $(1, x, \ldots, x^{m-1})$ of A_m. Let us consider the complex (1.11) with $m = 0$. It reduces to

$$0 \to 0 \to A_p \oplus A_q \xrightarrow{(P,Q)} A_{m+p+q} \to 0. \tag{1.16}$$

In the standard bases, the map (P, Q) is represented by the matrix

$$\begin{pmatrix} p_0 & 0 & 0 & \cdots & 0 & q_0 & 0 & 0 & \cdots & 0 \\ p_1 & p_0 & 0 & \cdots & 0 & q_1 & q_0 & 0 & \cdots & 0 \\ p_2 & p_1 & p_0 & \cdots & 0 & q_2 & q_1 & q_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where p_i, q_j are the coefficients of P, Q, therefore ϕ is the classical resultant, and ψ is equal to ϕ multiplied by $(-1)^{pq}$ for normalized P, Q. This recovers the calculation of [P].

It would be interesting to compare the proof of the Weil reciprocity law given here with that of Arbarello, De Concini, and Kac [ADCK].

2. NON-LINEAR FOURIER TRANSFORM

In this section we recall that to a function f belonging to the ring $A = \mathcal{O}(X \backslash \{a\})$ introduced in section 0 there is associated an ordinary differential operator (ODO for short) L_f, in such a way in fact that the order of pole of f at a equals the order of the ODO L_f. This way, the two sides of the reciprocity law equal (up to sign) the resultant of the corresponding ODO's, a concept completely analogous to the resultant of two polynomials, as noticed in [P].

We describe three aspects of the dictionary between functions and differential operators. The first is very explicit because of the choice of coordinates made. The second is a more intrinsic construction of it, due to

Burchnall and Chaundy in the 1920s and rediscovered by Krichever, cf. [Mum]. This gives, for a choice of line bundle $\xi \in \mathrm{Jac}X\backslash\Theta$, an algebra embedding

$$\phi_\xi : A = H^0(X\backslash\{a\}, \mathcal{O}_X) \to \mathcal{D}_0, \tag{2.1}$$

where $\mathcal{D}_0$ is the ring of ODO's with coefficients analytic in a neighborhood of $x = 0$, say. It contains the first in a formal way which forgets the choice of the line bundle. The third is yet more general, and generalizes the Krichever map to

$$\phi : H^0(J\backslash\Theta, \mathcal{O}_J) \to \mathcal{M}, \tag{2.2}$$

where J is an abelian variety, Θ a principal polarization for J, and $\mathcal{M}$ is a ring of matrix differential operators in several variables.

The point of bringing in the three versions of the dictionary, although only one would suffice in this paper, is that we want to propose a program in which the higher-dimensional reciprocity laws (cf., e.g. [BMcL]) can be translated into results on sets of commuting matrices of partial differential operators (PDO's).

2.1. *Without the Line Bundle*

We map in a completely formal way the ring A of Section 0 to the ring Ψ of formal pseudodifferential operators:

$$\left\{ \sum_{j=-\infty}^{N} u_j(x)\partial^j, u_j \text{ analytic in some connected neighborhood of } x = 0 \right\}. \tag{2.3}$$

The map $f \mapsto \mathcal{L}_f$ is given by the "Fourier transform"

$$\mathcal{L}_f e^{x\kappa} = f(\kappa)e^{x\kappa}, \tag{2.4}$$

where $\kappa = z^{-1}$.

Notice that the operators in the image of A have constant coefficients. Aside from obtaining a ring isomorphic to A, we have not gained anything; in the following Section 2.2 we will see how this ring can be made to flow in the x variable by introducing a line bundle over the curve.

2.2. *With the Line Bundle*

Let a be a point of the Riemann surface X of genus g, and $p_1, \dots, p_g$ points of X such that $h^0(\mathcal{O}(p_1 + \dots + p_g - a)) = 0$ and $h^1(\mathcal{O}(p_1 + \dots + p_g)) = 0$; to these data, and to the choice of a local parameter z near a as in section 0, or its inverse κ, we can associate an algebra embedding

$$\phi_\xi : A = H^0(X\backslash\{a\}, \mathcal{O}_X) \to \mathcal{D}_0, \tag{2.5}$$

where we denote by ξ the line bundle $\mathscr{O}(p_1 + \cdots + p_g)$. We refer to [Mum] for the details, but we sketch the construction because the main tool, the Baker–Akhiezer function introduced by Krichever, serves to link our three constructions.

DEFINITION 2.2.1. Under the above assumptions, there exists a unique analytic function $\psi(p, x)$ of $p \in X$ and $x \in \mathbb{C}$, whose domain is the product of $X \backslash \{a\}$ and a disc D in $\mathbb{C}$ centered at 0, such that ψ has poles exactly at $p_1, \ldots, p_g$ and an essential singularity at a depending on a (complex) parameter x near 0, with asymptotic expansion

$$\psi(\kappa, x) = e^{x\kappa}(1 + O(z)). \tag{2.6}$$

This is called a Baker–Akhiezer function.

Now the embedding can be defined as $f \mapsto L_f = L_f(x, \partial)$ where $L_f \psi = f(p)\psi(p, x)$. To explain this, we introduce the line bundle ξ_x over $X \times D$ obtained from ξ by multiplying by $e^{x\kappa}$ a trivialization on $X \backslash \{a\}$ and glueing to a fixed trivialization on a disc around a. The space of sections of the line bundle ξ_x over $X \times D$ is a free module of rank 1 over the algebra $\mathscr{O}(D)[\partial]$. Then the space of sections of ξ_x over $X \backslash \{a\} \times D$ which are meromorphic at a becomes a free module of rank 1 over $\mathscr{D}_0$, generated by $\psi e^{-x\kappa}$.

Note that under this embedding the order m of pole of the function at a is equal to the order of the corresponding ODO. The ODO has higher order term $a\partial^m$.

It was proved by Sato that this ring of ODO's can be conjugated by an element S of Ψ into a *constant-coefficient* subring of Ψ, which is the object we constructed in 2.1. This S encodes the line bundle, as it is related to the Baker–Akhiezer function by $\psi(x, z) = Se^{x\kappa}$ with the formal action

$$\partial^{-1} e^{x\kappa} = \kappa^{-1} e^{x\kappa}. \tag{2.7}$$

We can think of the pseudodifferential operator $\mathscr{L} = S\partial S^{-1}$ as playing the role of a local parameter inverse; indeed $\mathscr{L}$ acts on the Baker–Akhiezer function as $\mathscr{L}\psi = \kappa\psi$. This is a spectral transform or non-linear Fourier transform as will be explained in Section 2.3.

2.3. *The Nakayashiki Map*

Nakayashiki in [N1] and [N2] gave a dramatic generalization of the Krichever map to the case when X is a higher-dimensional variety and A is the ring of functions regular over X outside a suitable divisor; there are more restrictive conditions on the additional choices such as the line bundle. We define the Nakayashiki map only for the case that pertains to our situation, namely when $X = J$ is a Jacobian. Our future program is to extend, via this map when dim J =2, an interpretation of the two-dimensional reciprocity laws (see [BMcL]).

Let J be a g-dimensional principally polarized abelian variety with principal polarization Θ and with a choice of basis of first homology so that we can identify $J = \mathbb{C}^g/\mathbb{Z}^g + \Omega\mathbb{Z}^g = \mathrm{Pic}^0(J)$. The Poincaré bundle $\mathscr{P}$ is normalized in such a way that $\mathscr{P}|_{J\times\{c\}} \cong \mathscr{L}_c$, the line bundle defined by the cocycle $\rho(m+\Omega n, z) = \exp(2\pi i^t m \cdot c)$, for a vector $c \in \mathbb{C}^g$ (we use the same notation for its class modulo $\mathbb{Z}^g + \Omega\mathbb{Z}^g$). We also normalize a basis $\eta_1, \ldots, \eta_g$ of differentials of the second kind on J with poles on Θ only.

In the first instance, we give the Nakayashiki map associated to the trivial sheaf on J, which is very concrete; in [N2] Nakayashiki generalized it to a coherent sheaf F on J, satisfying certain technical conditions, and we state that result in the case which reduces to the situation of Section 2.2.

We let $\mathscr{F}(J,\Theta)$ be the Fourier–Mukai transform of $\mathscr{O}_J(*\Theta)$ [Muk], namely:

$$\mathscr{F}(J,\Theta) = \cup_n \mathscr{F}(J,\Theta)(n),$$

where $\mathscr{F}(J,\Theta)(n) = \pi_{2*}(\pi_1^*\mathscr{O}_J(n\Theta) \otimes_{\mathscr{O}_J\otimes\mathscr{O}_{\mathrm{Pic}^0(J)}} \mathscr{P})$. Here π_i are the projections of $J \times \mathrm{Pic}^0(J)$ to the ith factor. If is proved in [N1] that $\mathscr{F}(J,\Theta)$ is a coherent $\mathbf{D} := \mathbf{D}_{\mathrm{Pic}^0(J)}$-module over $\mathrm{Pic}^0(J)$, where $\mathbf{D}$ is the sheaf of algebraic differential operators on $\mathrm{Pic}^0(J)$. This is an elaboration of the Fourier transform of Laumon [L]. In the notation of [N1], let $\mathscr{D}_{\mathrm{Pic}^0(J)} = \mathbf{C}\{\{x_1,\ldots,x_g\}\}[\partial/\partial x_1,\ldots,\partial/\partial x_g]$, which is the algebra of differential operators defined on a neighborhood of 0 in $\mathrm{Pic}^0(J)$. For any $c \in \mathbb{C}^g$, the stalk of $\mathbf{D}$ at c is identified with $\mathscr{D}$. Finally, let F_c be the stalk of $\mathscr{F}(J,\Theta)$ at c.

Definition 2.3.1. A Baker–Akhiezer vector is an element of F_c.

Theorem 2.3.2. [*N*1, *Th*. 2.3]. *If Θ is smooth, F_c is a free $\mathscr{D}_{\mathrm{Pic}^0(J)}$-module of rank $g!$ for $c \neq 0$.*

This statement gives the desired embedding (which depends on c), as in Section 2.2, of the algebra of regular functions on $J\backslash\Theta$ into matrix-PDO's in g variables of size $g!$, by action on the Baker–Akhiezer vectors after a choice of basis. Indeed the algebra $\mathscr{O}(J,*\Theta)$ operates on the sheaf $\mathscr{O}_J(*\Theta)$, hence it operates on its Fourier–Mukai transform and on the stalk F_c. This operation is $\mathscr{D}$-linear, hence, if we choose a basis of F_c as a $\mathscr{D}$-module, it is realized by matrices with entries in $\mathscr{D}$.

There is also a more concrete interpretation of Baker–Akhiezer vectors in the spirit of Section 2.2: these are meromorphic functions of $(z,x) \in [J\backslash\Theta] \times U$, where $x = (x_1,\ldots,x_g)$ and U is an open set in $\mathrm{Pic}^0(J)$, which along $\Theta \times U$ behave like the product of a holomorphic function and of

$$\Psi(z,x) = \exp\left(-\sum_{i=1}^{g} x_i \int_0^z \eta_i\right). \tag{2.8}$$

The isomorphism between F_c and the space of such Baker–Akhiezer functions is given by multiplication by Ψ.

In [N2] the construction is extended to more general coherent shaves on J than the trivial sheaf. In particular, it is proved [N2, Sect. 5] that the same conclusion of Theorem 2.3.2 holds, without the assumption that Θ is smooth, for the sheaf $\mathcal{O}_X(na)$ supported on the curve, embedded in J via the Abel map, and twisted by a negative multiple na of a point $a \in X$. Moreover, in this case, the Fourier–Mukai transform of $\mathcal{O}_X(*a)$ is a sub-holonomic $\mathbf{D}$-module A, as follows from [R, Sects. 8 and 9]. This $\mathbf{D}$-module depends on some auxiliary choices. However, we can describe concretely its structure of module over a smaller sheaf of rings. To the pair (X, a) is attached a vector $v \in T_0(J)$ up to a scalar, namely the derivative at a of the Abel–Jacobi map. Then v determines an invariant vector field on J, which we also denote by v. Let $\mathbf{D_1} = \mathcal{O}_J[v] \subset \mathbf{D}$. Then we have

$$A = \mathbf{D_1}, \tag{2.9}$$

i.e., A is free of rank 1 as a $\mathbf{D}_1$-module. This result is proved in [R, Sect. 8]. In this case, the algebra of $\mathbf{D}$-linear endomorphisms of A is locally isomorphic to $\mathcal{O}_J[\partial/\partial x]$ [N2, Appendix]. Here x is a linear parameter along the trajectories of v. Thus in this case, taking the stalks at some c, we get a mapping from $H^0(X\backslash\{a\}, \mathcal{O}_X)$ to the ring $\mathcal{O}_J[d/dx]$. When we restrict to a translate of a suitable one-parameter subgroup of $J = \mathrm{Pic}^0(J)$, we recover the Krichever construction. The image of c in J is the line bundle.

We then have

$$A = \mathbf{D}/I. \tag{2.9}$$

In this case, the algebra of $\mathbf{D}$-linear endomorphisms of A is locally isomorphic to $\mathcal{O}_J[\partial/\partial x]$ [N2, Appendix]. Here x is a linear parameter such that the foliation is defined by $x =$ constant. Thus in this case, taking the stalks at some c, we get a mapping from $O(X, *a)$ to the ring $\mathcal{O}_J[d/dx]$. When we restrict to a translate of a suitable one-parameter subgroup of $J = \mathrm{Pic}^0(J)$, we recover the Krichever construction. The image of c in J is the line bundle.

3. THE TRANSFORM OF THE KOSZUL COMPLEX

In this section, we use the dictionary of Section 2 to translate the reciprocity law into an equality of determinants of matrices of functions of x. These are the matrices of the action of the differential operators, associated to functions on the curve, on certain finite-dimensional vector spaces. This completes the construction given in [P] by translating the multiplication by a function into an action of the associated operator and thus allowing for the double complex of Section 1 to yield the proof.

The differential operators L_f and L_g are holomorphic in some disc D in the x-plane centered at 0. Denote by E_g the space of germs of holomorphic functions at 0 annihilated by L_g; the dimension of E_g is equal to m, since the leading term of L_g does not vanish at 0.

LEMMA 3.1. *There is a non-degenerate pairing*

$$(A/gA) \times E_g \to \mathbb{C} \tag{3.1}$$

given by

$$(h \bmod g, \phi) \mapsto [L_h\phi](0). \tag{3.2}$$

Proof Note that A/gA also has dimension n. So the non-degeneracy follows if we prove that for $h \in A$ such that L_h annihilates E_g, then h belongs to gA. We can assume that h has minimal order of pole r with that property. Then r is not divisible by n, otherwise we could add to h a multiple of a power of g to decrease its order. Now we can write $L_h = PL_g + Q$ where P, Q are differential operators with Q of order $< n$. Since n does not divide r, Q cannot be 0. So the operator Q is of order $\leq n-1$ and annihilates the space of functions E_g of dimension n, which is a contradiction. ∎

Clearly with respect to the perfect duality (3.1), the adjoint of multiplication by f on A/gA is given by the operator L_f on E_g. This shows that the determinant of f on A/gA is equal to the determinant of L_f acting on the space of solutions E_g, which is shown in [P] to be the same as the "differential resultant" of the two ODO's L_f and L_g. This shows the relation between the Weil reciprocity law and the reciprocity law involving the L_f proved in [P, Sect. 1]. This relation was observed in [P, Sect. 3] in a different language.

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